Neurons and neural networks I.

High-level vs. low level modeling

Previously in this course:

* Mechanistic description of neurons
* Descriptive models of neuronal operations

Goals of the next classes:

Are these models compatible with the neural architecture?
How can neural computations interpreted in this framework?

Outline

- single neuron
- neural network

- supervised learning
- unsupervised learning

Neural networks, basic concepts

Goal of the neural network can be phrased:

learning about its input
forming memories

Address based memory

- not associative
- not fault tolerant
- not distributed

Biological memory

- associative
- error tolerant
  - cue error
  - hardware fault
- parallel and distributed

Terminology

Architecture: what variables are involved, how they interact (weights, activities)

Activity rule: short time-scale behavior
how activitiy of neurons depends on inputs

Learning rule: long time-scle behavior
how weights change as a function of activities of other neurons, time, or other factors

Terminology

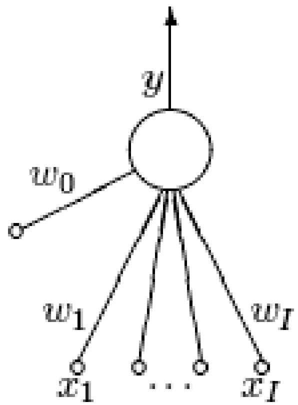
Supervised learning: *labelled data* is provided, i.e.
an input-output relationship is given;
the ‘teacher’ tells whether the response of
the neuron to a given input was correct

Unsupervised learning: *unlabelled data* is available;
learns the statistics of input:
generating a table of occurences

some more sophisticated *representation*

is learned

Single neuron as a classifier
Definitions



Architecture: I inputs x_i

weights w_i

Activity rule:

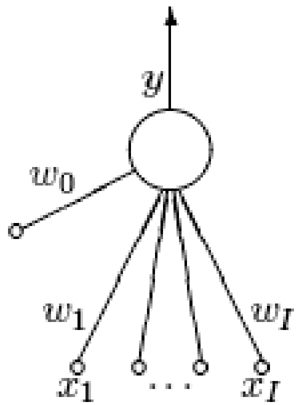
1. activation is calculated

$$a = \sum_i w_i x_i,$$

2. output is calculated
   - linear $\quad y(a) = a.$
   - logistic $\quad y(a) = \frac{1}{1+e^{-a}}$
   - tanh $\quad y(a) = \tanh(a)$
   - threshold

$$y(a) = \Theta(a) \equiv \begin{cases} 1 & a > 0 \\ -1 & a \leq 0. \end{cases}$$

Single neuron as a classifier Definitions



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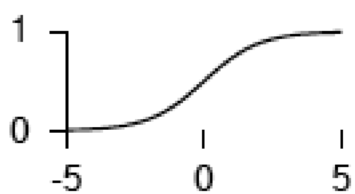

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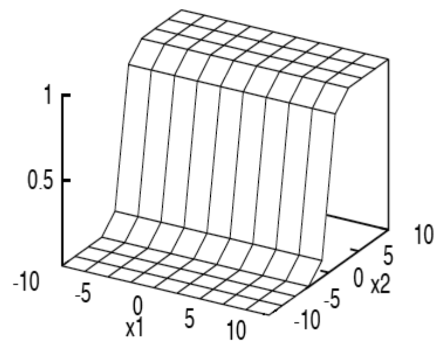
Single neuron as a classifier
Concepts

Input space

$$y(\mathbf{x};\mathbf{w}) = \frac{1}{1+e^{-(w_1 x_1 + w_2 x_2)}}$$



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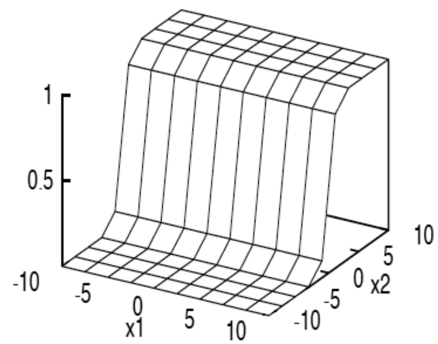

$\mathbf{w} = (0, 2)$

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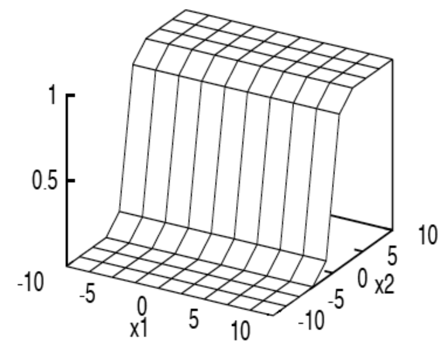

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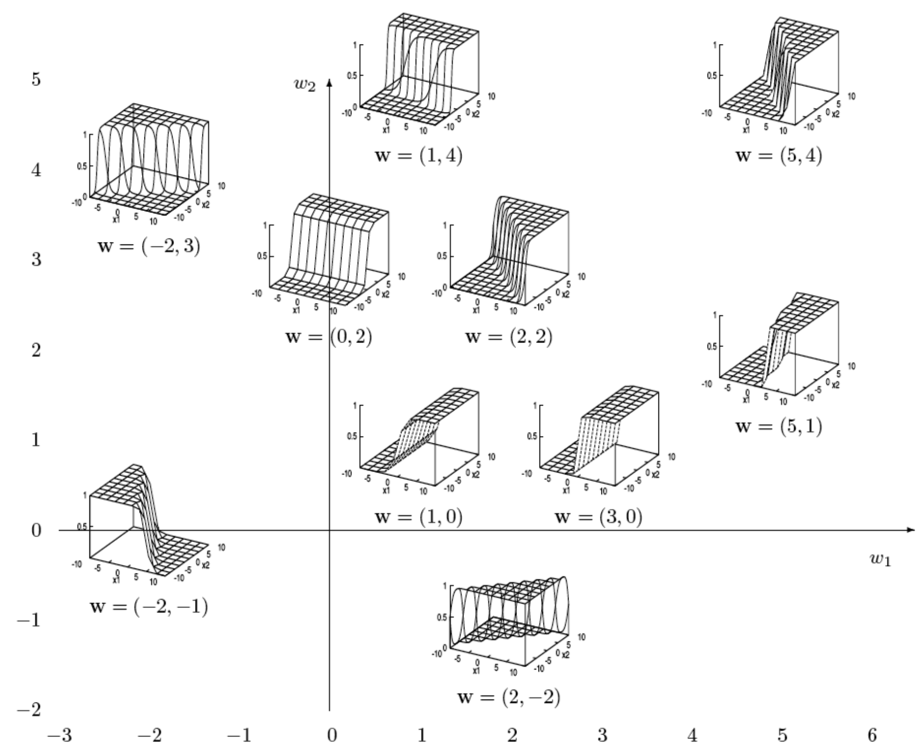
Weight space



Single neuron as a classifier
Concepts

The supervised learning paradigm:

given example inputs $\mathbf{x}$ and target outputs t learning the mapping between them

the trained network is supposed to give ‘correct response’ for *any given* input stimulus

training is equivalent to learning the appropriate weights

in order to achieve this, an *objective function* (or *error function*) is defined, which is minimized during training

Binary classification in a neuron

Error function:

$$G(\mathbf{w}) = -\sum_{n} \left[t^{(n)} \ln y(\mathbf{x}^{(n)}; \mathbf{w}) + (1 - t^{(n)}) \ln(1 - y(\mathbf{x}^{(n)}; \mathbf{w})) \right]$$

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Learning: back-propagation algorithm

$$g_j = \frac{\partial G}{\partial w_j} = \sum_{n=1}^{N} -(t^{(n)} - y^{(n)})x_j^{(n)}$$

Here $e^{(n)} \equiv t^{(n)} - y^{(n)}$ is the error

Binary classification
The algorithm

1. Get the data
2. Calculate activations: $a = \sum_i w_i x_i,$
3. Calculate the output of the neuron: $y(a) = \dfrac{1}{1 + e^{-a}}$
4. Calculate error: $e = t - y$
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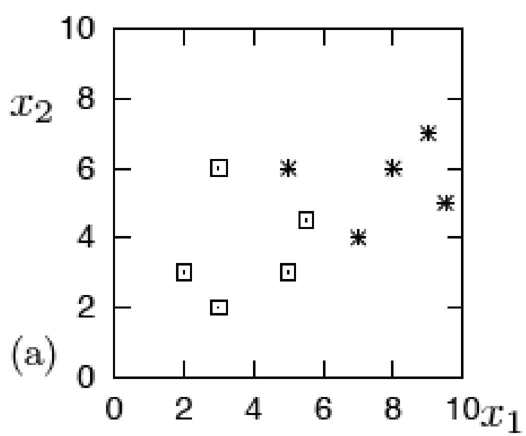

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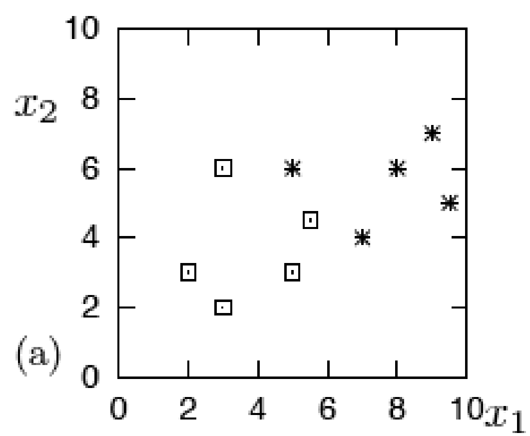

gradient descent

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stochastic gradient descent

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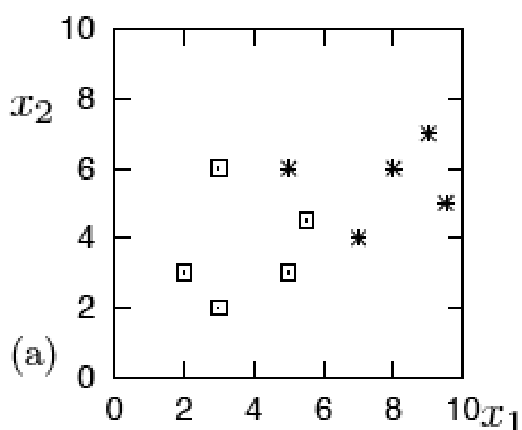
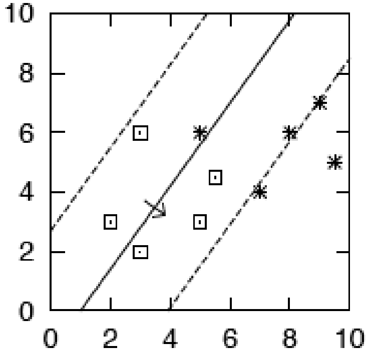


gradient descent

Binary classification Demonstration



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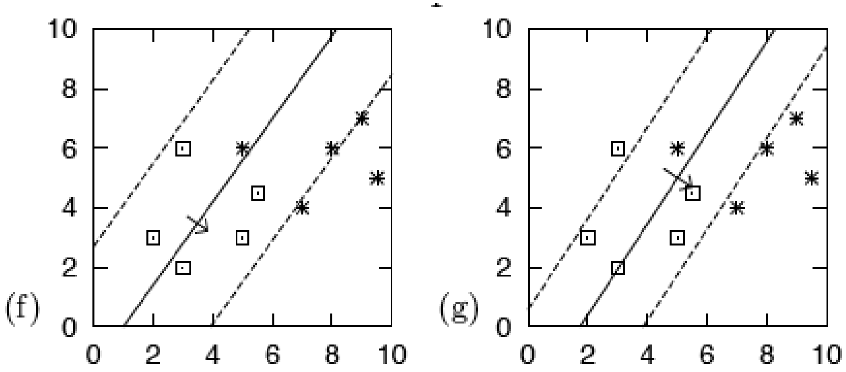
Binary classification Demonstration



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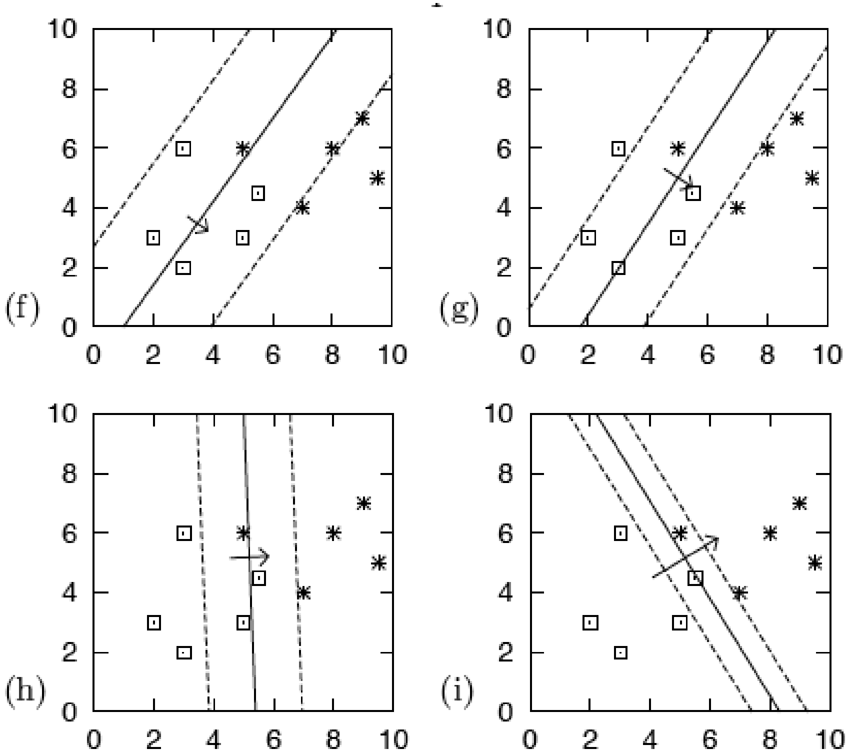

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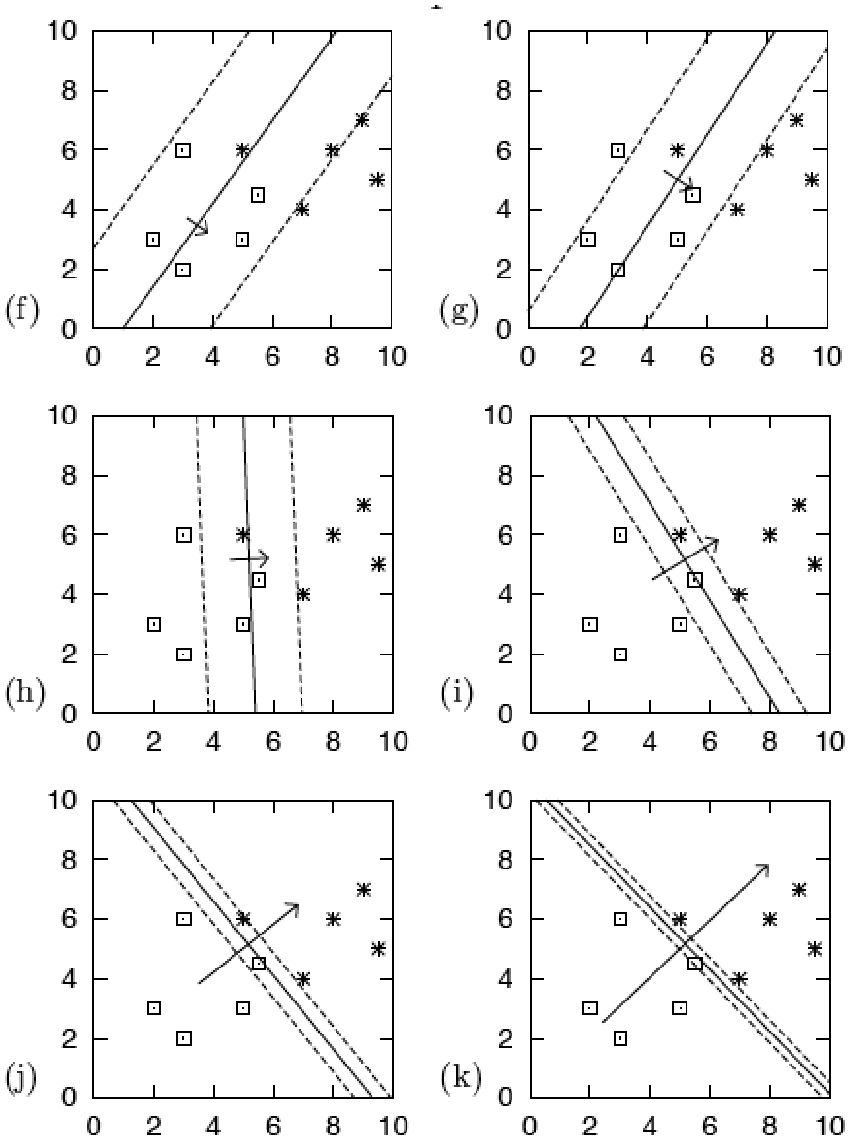

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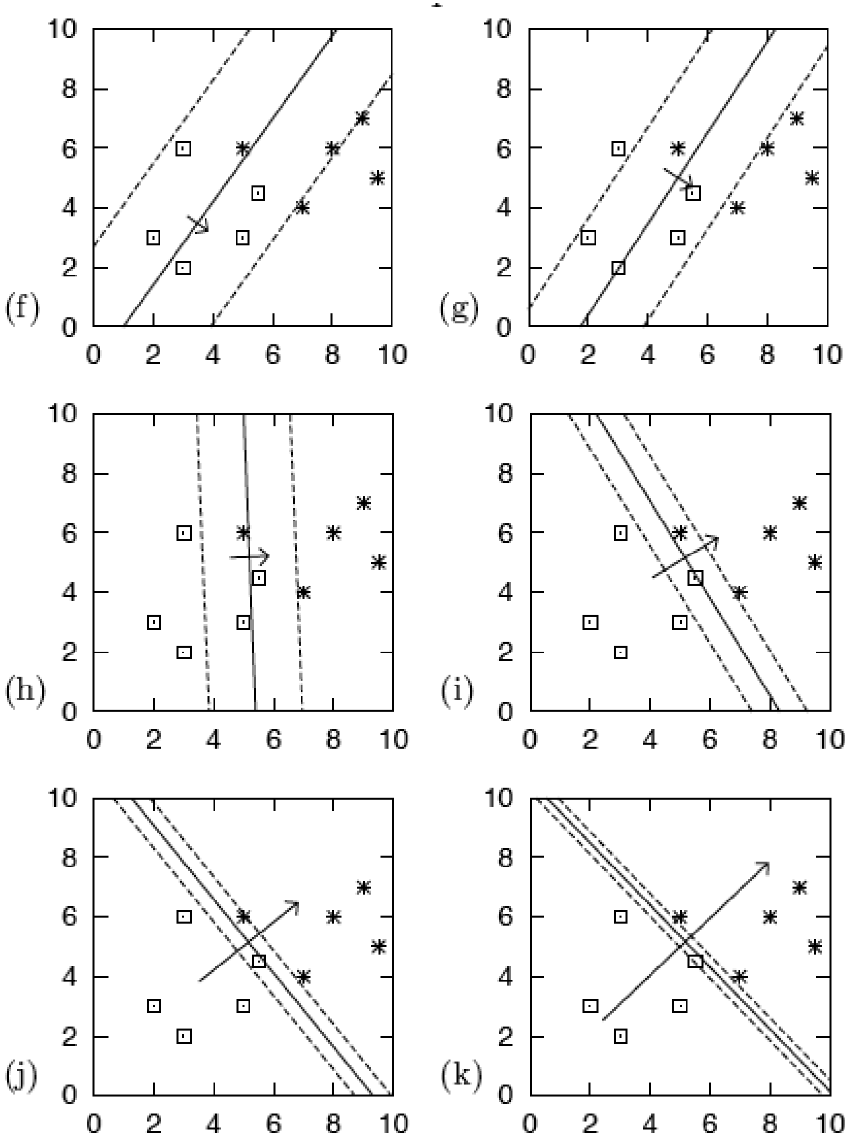
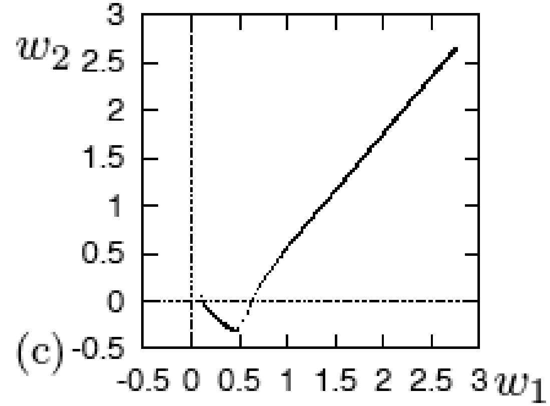


Evolution of weights



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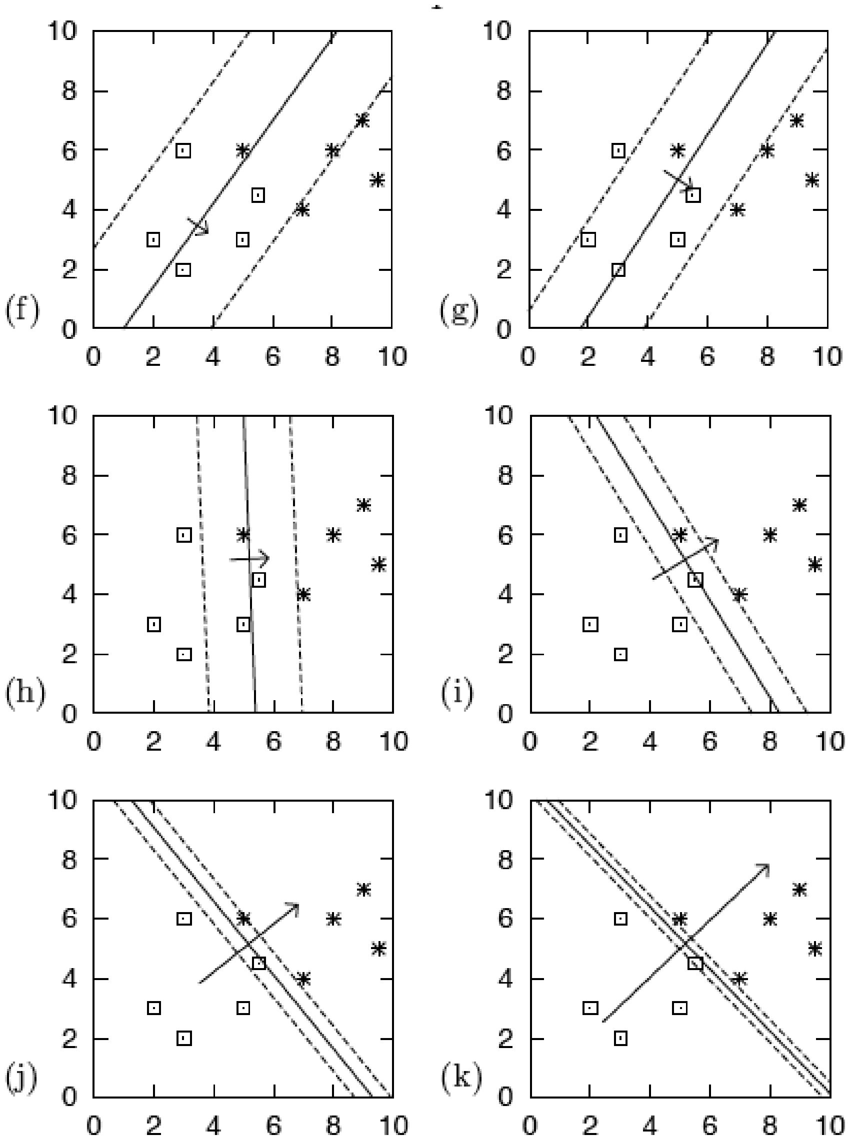

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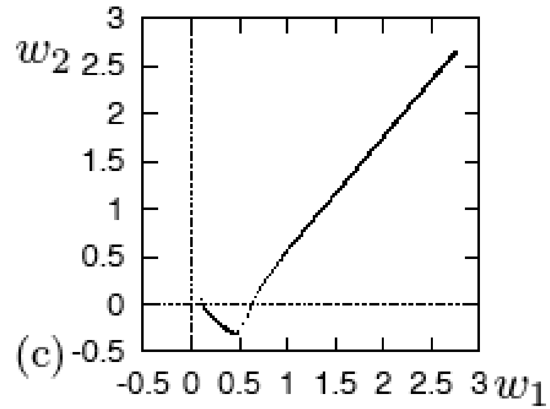
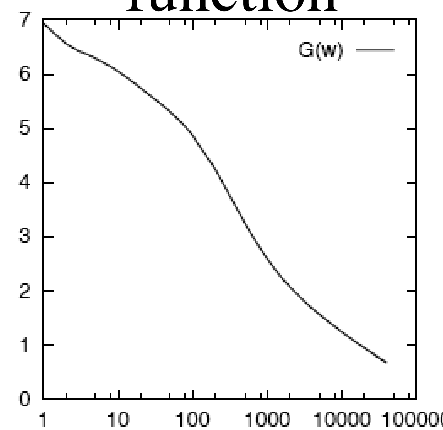
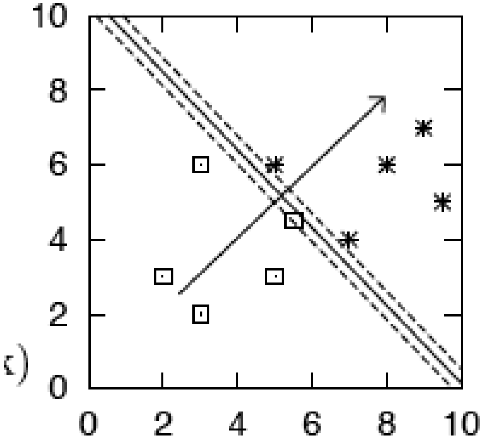


Evolution of the error function



Binary classification
Are we happy?



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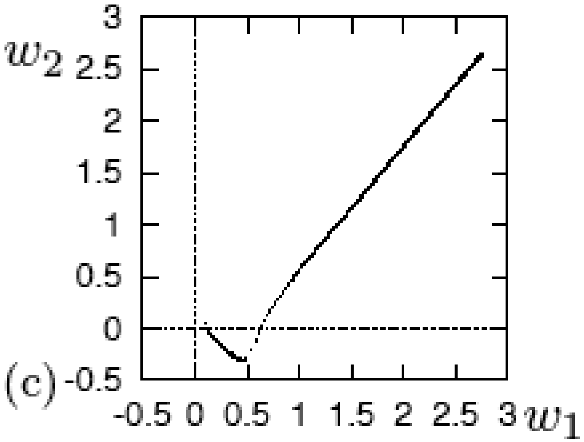

w's seem to grow unbounded

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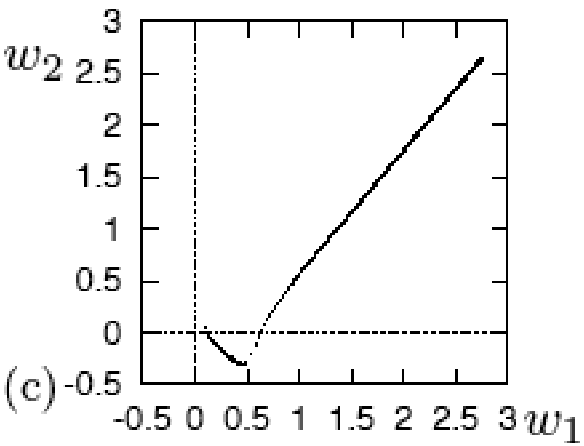

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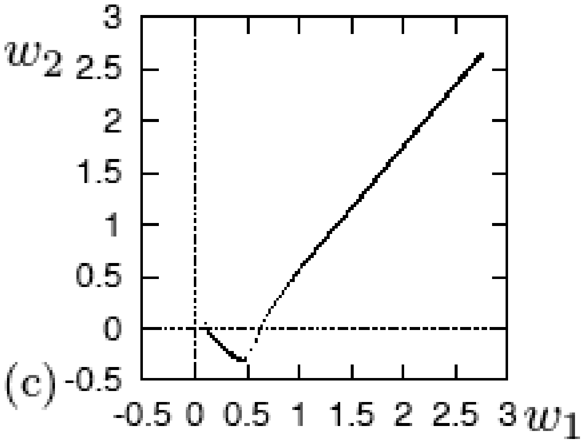

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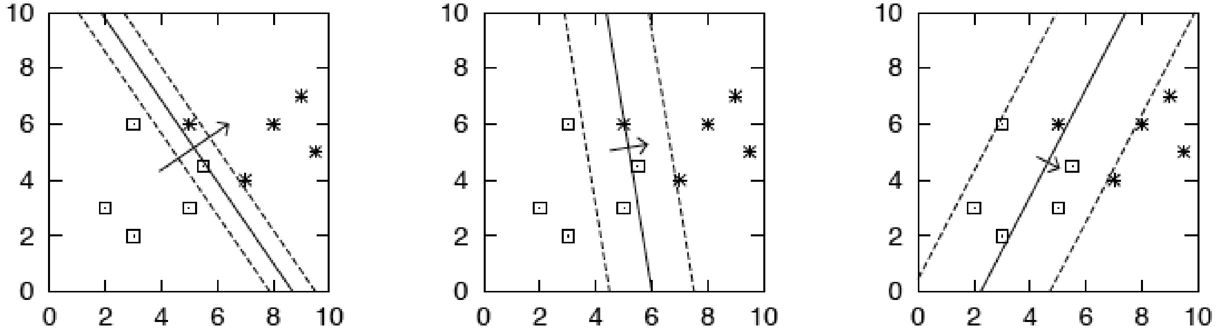
Generic pain killer: *regularization*

$$M(\mathbf{w}) = G(\mathbf{w}) + \alpha E_W(\mathbf{w})$$

$$E_W(\mathbf{w}) = \frac{1}{2}\sum_i w_i^2. \quad \text{(weight decay)}$$

Binary classification
Effect of regularization



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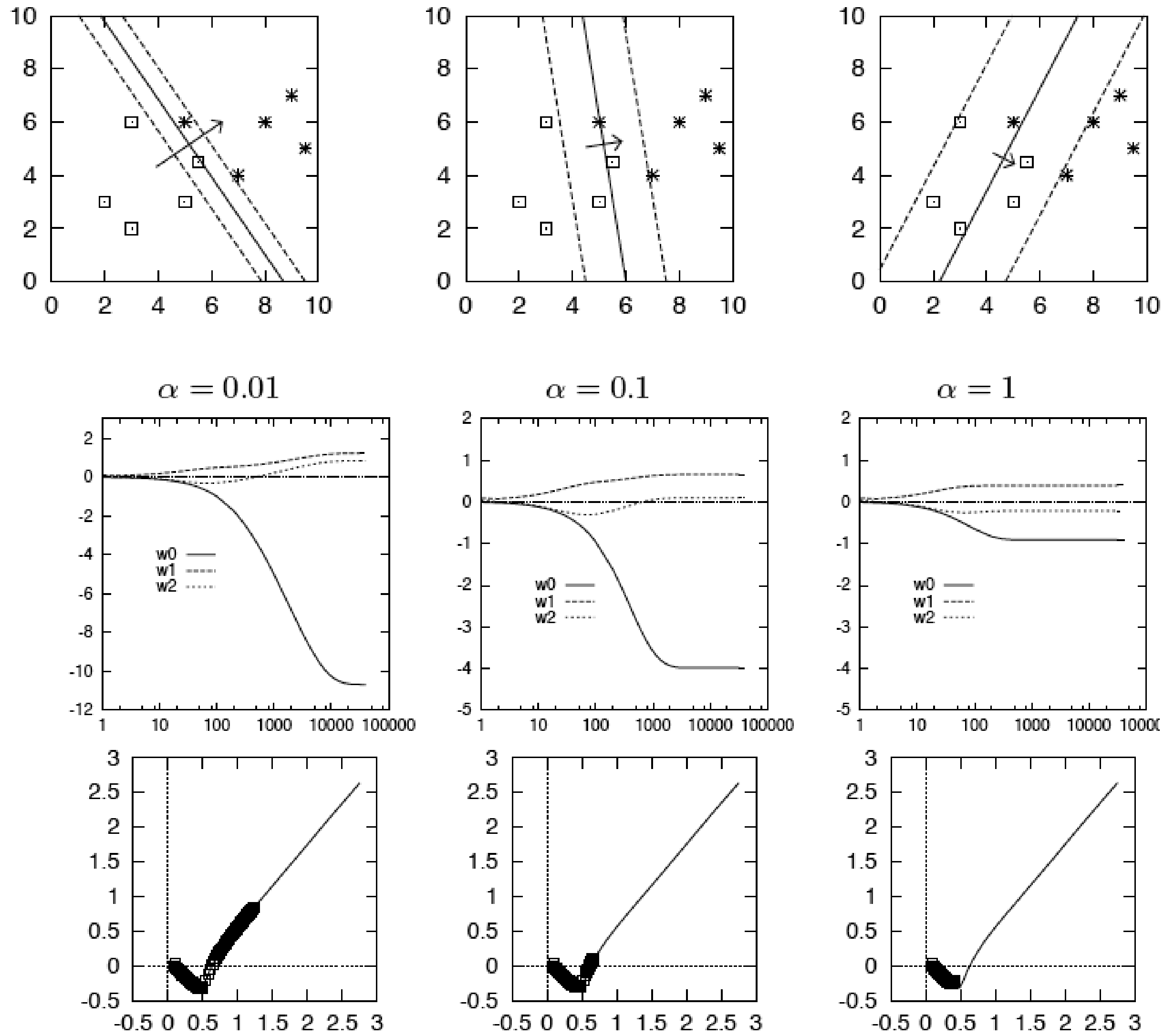

Capacity of the neuron

$$
\begin{tikzcd}
\{t_n\}_{n=1}^{N} \arrow[r] & \begin{array}{c}\text{Learning}\\ \text{algorithm}\end{array} \arrow[r] & \mathbf{w} \arrow[r] & \mathbf{w} \arrow[r] & \{\hat{t}_n\}_{n=1}^{N} \\
& \{\mathbf{x}_n\}_{n=1}^{N} \arrow[u] & & \{\mathbf{x}_n\}_{n=1}^{N} \arrow[u] &
\end{tikzcd}
$$

How much information can the neuron transmit?

Interpreting learning as inference

So far: optimization wrt. an objective function

$$M(\mathbf{w}) = G(\mathbf{w}) + \alpha E_W(\mathbf{w})$$

where

$$G(\mathbf{w}) = -\sum_n \left[t^{(n)} \ln y(\mathbf{x}^{(n)}; \mathbf{w}) + (1 - t^{(n)}) \ln(1 - y(\mathbf{x}^{(n)}; \mathbf{w})) \right]$$

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What's this quirky regularizer, anyway?

Interpreting learning as inference

Let's interpret $y(\mathbf{x},\mathbf{w})$ as a probability:

$$\begin{aligned} P(t=1 \mid \mathbf{w}, \mathbf{x}) &= y \\ P(t=0 \mid \mathbf{w}, \mathbf{x}) &= 1-y \end{aligned}$$

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in a compact form:

$$P(t\,|\,\mathrm{w},\mathrm{x}) = y^t(1-y)^{1-t} = \exp[t\ln y + (1-t)\ln(1-y)]$$

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$\Longrightarrow$ the regularizer has the form of a prior!

$$P(\mathbf{w} \,|\, \alpha) = \frac{1}{Z_W(\alpha)} \exp(-\alpha E_W).$$

Interpreting learning as inference

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what we get in the objective function M(**w**):

the posterior distribution of **w**: $P(\mathbf{w}\,|\,D,\alpha) = \dfrac{P(D\,|\,\mathbf{w})P(\mathbf{w}\,|\,\alpha)}{P(D\,|\,\alpha)}$

Interpreting learning as inference Relationship between $M(\mathbf{w})$ and the posterior

$$\begin{aligned} P(\mathbf{w} \mid D, \alpha) &= \frac{P(D \mid \mathbf{w}) P(\mathbf{w} \mid \alpha)}{P(D \mid \alpha)} \\ &= \frac{e^{-G(\mathbf{w})} \, e^{-\alpha E_W(\mathbf{w})} / Z_W(\alpha)}{P(D \mid \alpha)} \\ &= \frac{1}{Z_M} \exp(-M(\mathbf{w})). \end{aligned}$$

interpretation: minimizing $M(\mathbf{w})$ leads to finding the *maximum a posteriori* estimate $\mathbf{w}_{\mathbf{MP}}$

The log probability interpretation of the objective function retains:

additivity of errors, while

keeping the multiplicativity of probabilities

Interpreting learning as inference
Properties of the Bayesian estimate

Interpreting learning as inference
Properties of the Bayesian estimate

- The probabilistic interpretation makes our assumptions explicit:

  by the regularizer we imposed a soft constraint on the learned parameters, which expresses our *prior expecations*.

- An additional plus:

  beyond getting $\mathbf{w}_{\mathbf{MP}}$ we get a measure for learned parameter uncertainty