Causality and manifold dimensions

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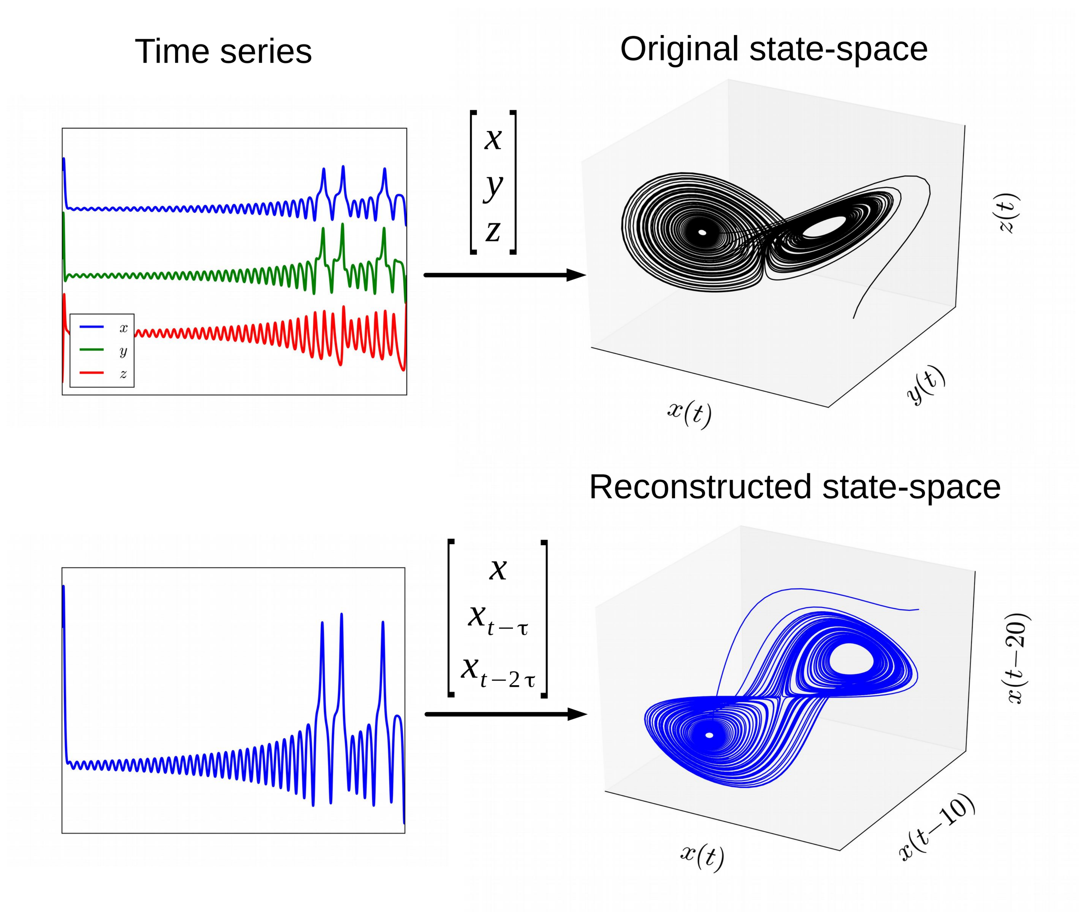
Abstract

Detection of causal relation from time series is a delicate task and no methods existed yet, which are able to distinguish, whether the variables under investigation are unidirectionally or bidirectionally causally linked, share a common cause or independent of each other.
Mutual Information Dimension is a known symmetric measure of interdependence which was employed to reveal nonlinear linkage between random variables. We extended this measure by taking into consideration the symmetry or asymmetry in the reconstructed manifold dimensions and showed, that the new method makes causal inference possible, and potentially able to distinguish the four cases above.
We present our method in work on simulation examples and show preliminary applications to neuro-electrophysiological data.

This research was supported by the Hungarian Scientific Research Found (OTKA) K-113147 and the Hungarian Brain Research Program - Grant No. KTIA_13_NAP-A-IV/1,2,3,4,6.

Takens theorem

If we have time series measurement from a deterministic dynamical system, we could reconstruct the state-space of the system with time delay embedding according to Takens theorem. The theorem also ensures that the dimension of the manifold in the reconstructed state-space is the same as it is in the original state-space.



Dimension

The dimension of an object is the following limit expression:

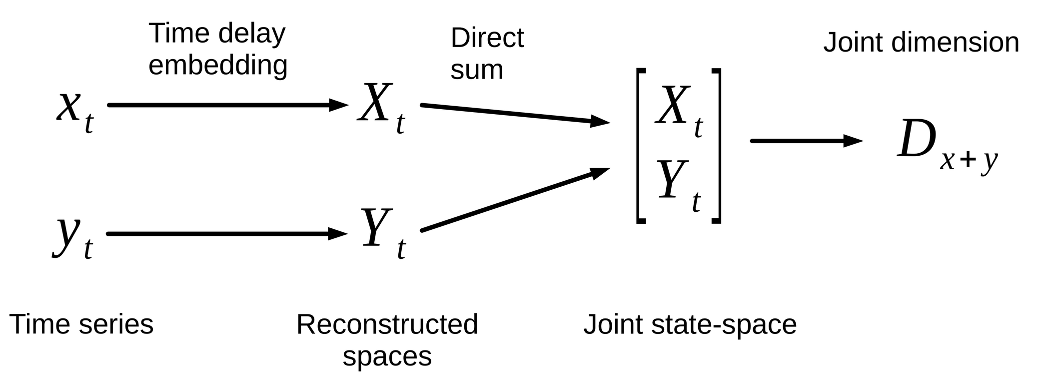
$$D_q = \lim_{\epsilon \to 0} \frac{1}{1-q} \frac{\log \sum_i p(\epsilon)_i^q}{\log \frac{1}{\epsilon}}$$

Depending on the value of q it can be the box-counting dimension (q=0), the information dimension (q→1) or the correlation dimension (q=2).
Non-rigourously the intrinsic dimension of a data-set is the minimum number of variables needed to describe it without losing significant information. In this work we used a maximum likelihood intrinsic dimension estimator (proposed by Levina and Bickel, enhanced by MacKay and Ghahramani):

$$D(k) = \frac{n(k-1)}{\sum_{i=1}^{n} \sum_{j=1}^{k-1} \log \left(\frac{T_k(x_i)}{T_j(x_i)} \right)}$$

Mutual dimension (MD)

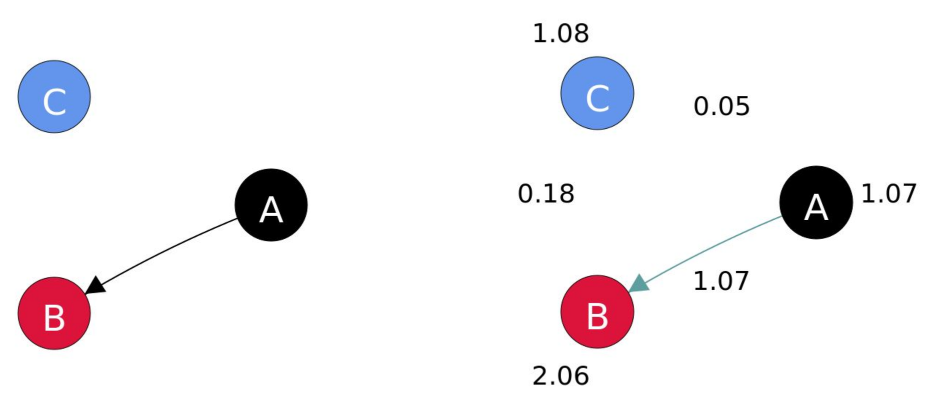
Mutual dimension is a nonlinear symmetric measure of interdependence. Given two simultaneous time series measurements, one can time delay embed both series separately and measure the intrinsic dimension of each point-cloud. One can get the joint dimension of the two state-space by forming the direct sum of the two spaces and measure the intrinsic dimension of the resulting point-cloud.



The *mutual dimension* is the difference of the sum of single dimensions and the joint dimension.

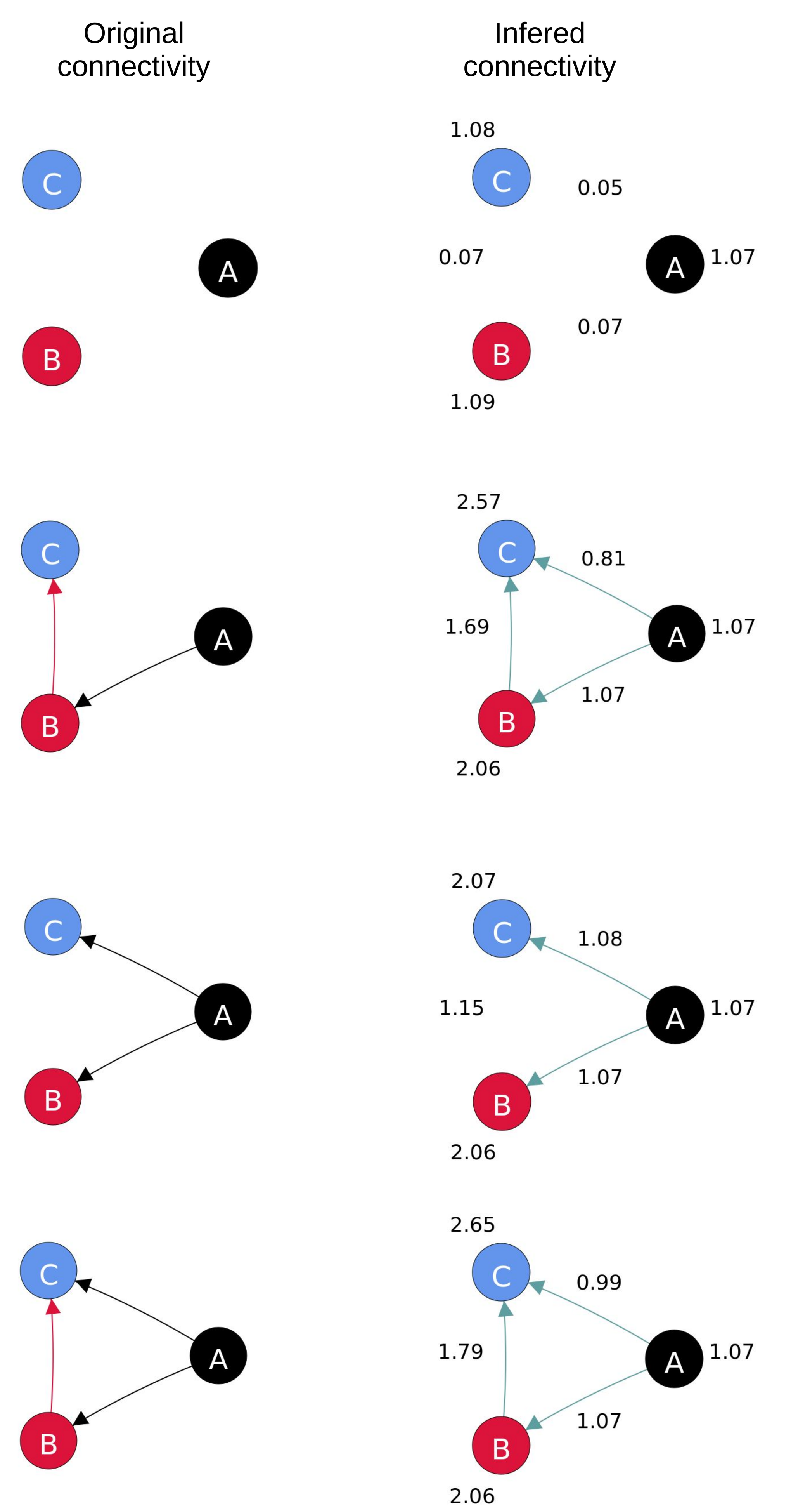
$$MD_{XY} = D_X + D_Y - D_{X+Y}$$

When the variables are independent, the mutual dimension is zero, so the joint dimension equals to the sum of the two single dimensions. When the variables are not independent, then mutual dimension is greater than zero.



Results

We simulated three coupled logistic maps (*A*,*B*,*C*) with different connectivities and calculated the intrinsic dimensions and pairwise mutual dimension for each reconstructed case.
We could reconstruct the transitive closure of the original connectivities.



Inferring causal relation from manifold dimensions

Independent case

$$X \perp Y \Rightarrow D_{X+Y} = D_Y + D_Y \rightarrow MD_{XY} = 0$$

Unidirectional case

$$X \rightarrow Y \Rightarrow D_{X+Y} = D_Y < D_X + D_Y \rightarrow MD_{XY} = D_X$$

Circular case

$$X \leftrightarrow Y \Rightarrow D_{X+Y} = D_X = D_Y < D_X + D_Y \rightarrow MD_{XY} = D_X = D_Y$$

Common cause

$$X - Y \Rightarrow D_{X+Y} < D_X + D_Y \rightarrow MD_{XY} < D_X, MD_{XY} < D_Y$$

Model system

The basic model system was a network of three logistic maps. The parameters were set that the system was operating in the chaotic regime.

$$A_{t+1} = r A_t (1 - A_t + \beta B_t + \gamma C_t)$$